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THE $T(4)$ PROPERTY OF FAMILIES OF UNIT DISKS

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ABSTRACT

For a family $\mathcal{F}$ of n mutually disjoint unit disks in the plane, we show that if any four disks are intersected by a line then there is a line that intersects at least $n-1$ disks of $\mathcal{F}$.

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1. Introduction

Let $\mathcal{F}$ denote a family of **ovals** (compact convex sets with non-empty interior) in the Euclidean plane. $\mathcal{F}$ has a **transversal** and the property T, if there exists a line that intersects all members of $\mathcal{F}$. If there is a line that meets all but at most k members of $\mathcal{F}$, then $\mathcal{F}$ has the property $T - k$. Next, if each n-element subfamily of $\mathcal{F}$ has a transversal, then $\mathcal{F}$ has property $T(n)$. Finally, an m**-transversal** of $\mathcal{F}$ is a line that meets at least m elements of $\mathcal{F}$. An m-transversal is **separating** if there are elements of $\mathcal{F}$ in each of the open half-planes determined by it.

A central problem of Transversal Theory is to determine for a given $\mathcal{F}$, the smallest n such that $T(n)$ implies T. We recall briefly the history of the problem and refer to [4], [8], [10], [15] and [18] for a more detailed review.

A. (L. Santaló [23], 1940). There is a finite family $\mathcal{F}$ of not mutually disjoint ovals such that $T(n)$ does not imply T for any $n < |\mathcal{F}|$.
B. (H. Hadwiger [13], 1956). If $\mathcal{F}$ is an infinite family of mutually disjoint congruent ovals, then $T(3)$ implies T.

Henceforth, we assume that $\mathcal{F}$ is finite and that the ovals of $\mathcal{F}$ are mutually disjoint.

C. (B. Grünbaum [11], 1958). There is a family $\mathcal{F}$ of congruent ovals such that $T(5)$ does not imply T.
D. (H. Tverberg [25], 1989). If $\mathcal{F}$ is a family of translates of an oval, then $T(5)$ implies T.
E. (B. Aronov, J. Goodman, R. Pollack and R. Wenger [2], 2000). There is a family $\mathcal{F}$ of unit disks such that $T(4)$ does not imply T.
F. (M. Katchalski, T. Lewis [22], 1980). If $\mathcal{F}$ is a family of translates of an oval, then $T(3)$ implies $T - k$ for some universal constant k.
G. (A. Bezdek [3], 1991). There is a family $\mathcal{F}$ of unit disks such that $T(3)$ does not imply $T - 1$.
H. (T. Kaiser [21], 2002). If $\mathcal{F}$ is a family of unit disks, then $T(3)$ implies $T - 12$.
I. (A. Heppes [16], 2004). If $\mathcal{F}$ is a family of unit disks, then $T(3)$ implies $T - 2$.

Although in this article we concentrate on families of unit disks, we mention that it is widely believed that $T(4) \Rightarrow T - 1$ for families $\mathcal{F}(K)$ of pairwise disjoint translates of an arbitrary oval K. The following argument shows that if there exists a counter-example to this conjecture, then there is one with at most twelve elements. The original argument is described in [24] for the case when $\mathcal{F}$ has the property $T(3)$ and $T-3$ but not $T-2$; and it also appears in [18].

Suppose that the family $\mathcal{F}(K)$ has properties $T(4)$ and $T-2$ but not $T-1$ and also assume that $\mathcal{F}$ is minimal. We are going to construct a hypergraph G whose vertices correspond to the members of $\mathcal{F}$. To every 5-element subset of $\mathcal{F}(K)$, which does not have property T, associate an edge of G. In this way, we obtain a 5-uniform hypergraph. By Tverberg's theorem, there is at least one 5-element subset of $\mathcal{F}$ which does not have property T, so G has at least one edge. The **transversal number** of a hypergraph is the minimum number of vertices that intersect all edges. In the case of G, the transversal number is clearly two because $\mathcal{F}$ has the property $T-2$. It is a known fact in extremal graph theory that a 5-uniform hypergraph with transversal number two has an induced subgraph G' on at most twelve vertices with transversal number two. Such a G' induces a subfamily of $\mathcal{F}(K)$ which has at most twelve members, has properties $T(4)$ and $T-2$ but not $T-1$. Such a subfamily would clearly be a counterexample for the conjecture.

Furthermore, if $\mathcal{F}(K)$ is a family of pairwise disjoint translates of an oval K which has property $T(4)$, then it has property $T-12$. This statement is the consequence of the following two results proved by Eckhoff and that are of special interest. We note that Theorem 1 deals with any family of ovals and not only with families of disjoint translates, and that Theorem 2 is used throughout the rest of the paper.

Let c denote the smallest positive integer such that $\mathcal{F}(K)$ may be partitioned into c subfamilies such that each subfamily has property T. In this case we say that $\mathcal{F}(K)$ has property T^c.

Theorem 1 (Eckhoff [9], 1973): *For any family of ovals in the plane, $T(4) \Rightarrow T^2$. Moreover, one may either choose the two lines to be orthogonal or one may specify the direction of one of the lines.*

Theorem 2 (Eckhoff [9], 1973): *For any oval $K \subset \mathbb{R}^2$ and family $\mathcal{F}(K)$, we have*

$$T(3) \Rightarrow T^2.$$

Moreover, the two partial transversals may be chosen to be parallel and not farther apart than the relative width of K in their common direction.

We denote points of the plane by $a, b, \ldots$ and all other subsets by $A, B, \ldots$. As before, the convex hull of $A \cup B$ is denoted by $[A, B]$, and for two disjoint disks, the **tangential separators** are the two common tangents that separate them.

By Theorem 2, there exist disks $W \neq Z$ in $\mathcal{F}(K)$ such that $U \cap [W, Z] \neq \emptyset$ for any $U \in \mathcal{F} \setminus \{W, Z\}$. Let S_1 and S_2 denote the two supporting lines of $[W, Z]$ that meet both W and Z.

Since in this article we deal with families of unit disks, the reader should have the following figure in mind.

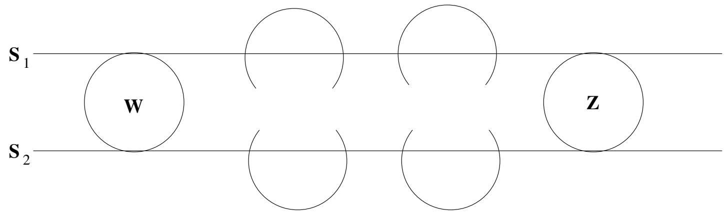

Figure 1

The following argument is due to Holmsen [17, 18]. Let $\mathcal{F}(K)$ denote a family of ovals that has property $T(4)$, and let L_1 and L_2 be the two partial transversal lines which intersect all members of $\mathcal{F}(K)$. The existence of L_1 and L_2 follows from Eckhoff's results. Let S be a strip of width at most $3\,\mathrm{diam}(K)$ that contains all members of $\mathcal{F}(K)$. We may further assume that L_2 is orthogonal to S. Then any translate of K which intersects L_2 is contained in a parallel strip, of width $2\,\mathrm{diam}(K)$ and centred at L_2. Therefore, all translates of K which are disjoint from L_1, are in $S \cap T$ whose area is at most $6\,\mathrm{diam}^2(K)$. With a suitable affine transformation, one obtains that the area of K is at least $1/2\,\mathrm{diam}^2(K)$. This implies that L_1 misses at most 12 elements of $\mathcal{F}(K)$.

Similar questions arise naturally for higher dimensions. For brevity, we outline only the most recent developments. Higher dimensional generalisations for families of balls were initiated by Problem 107 [14] of Hadwiger, which states that for any family of **thinly distributed** balls in $\mathbb{R}^n$, the property $T(n^2)$ implies T. A family of balls is **thinly distributed** if the distance between the centres of any two balls is at least twice the sum of their radii. Grünbaum [12] used the Topological Helly Theorem to prove that $T(2n-1)$ implies T under the same conditions. Ambrus, Bezdek and Fodor [1] recently showed that $T(n^2)$ implies T when the mutual distances of the centres of the n-dimensional unit balls are at least $2\sqrt{2+\sqrt{2}}$. Holmsen, Katchalski and Lewis [19] proved that there exists a positive integer $n_0 \leq 46$ such that $T(n_0)$ implies T for any family of pairwise disjoint unit balls in $\mathbb{R}^3$. This bound was later improved by Cheong, Goaoc and Na [7] to 18. Finally, Cheong, Goaoc, Holmsen and Petitjean [6] established that $T(4n-1) \Rightarrow T$ for any family of pairwise disjoint unit balls in $\mathbb{R}^n$. Holmsen and Matoušek [20] have recently shown that there is no such Helly-number for families of pairwise disjoint translates of an arbitrary convex body in $\mathbb{R}^3$. This indicates that the results above do not extend to translates of convex bodies in $\mathbb{R}^n$.

Henceforth, let $\mathcal{F}$ be a finite family of mutually disjoint unit disks with the property $T(4)$. In view of the above, it remains to determine if $\mathcal{F}$ has the property $T-1$.

2. Earlier Results

It has been proved in [4] and [5] that $\mathcal{F}$ has the property $T-1$ in the case $|\mathcal{F}| \leq 9$. In this section we outline the arguments described in these articles to help understand the general proof.

Let $n = |\mathcal{F}| \leq 7$. Consider $\mathcal{F} = \{W, A, B, C, D, E, Z\}$ with, say, A, B meeting S_2 and C, D, E meeting S_1, and the understanding that E is missing in the case $|\mathcal{F}| = 6$. In the latter case, assume that M is a transversal for $\mathcal{F} \setminus \{A, C\}$ and N is a transversal for $\mathcal{F} \setminus \{B, D\}$. If N is not a 5-transversal, then N strictly separates B and D. If a line strictly separates A and C, then it intersects N between $N \cap A$ and $N \cap C$, and as a consequence, it is disjoint form B or D. Thus, M meets A or C. The proof, when $n = 7$, requires a detailed case analysis. The basic assumption is that H is a transversal of $\{W, C, D, E, Z\}$, which is disjoint from A and B and is in limit position with respect to A in such a way that the distance between H and A is a minimum. The extreme property of H and the relative positions of $H \cap (W \cup C \cup D \cup E \cup Z)$ yield the existence of two sets U and V in $\mathcal{F} \setminus \{A, B\}$ that may be tangential to H in two different ways. The first is when H is a tangential separator of U and V, and the second is when H tangentially supports $[U, V]$. A delicate analysis (using the property

$T(4)$, the angles between tangent lines and the geometric properties of disks) yields that $\mathcal{F}$ has a 7-transversal.

Next, we assume $n = |\mathcal{F}| \geq 8$ and proceed by induction. From [4], we observe that $\mathcal{F}$ has the property $T - 1$ if at most two disks of $\mathcal{F} \setminus \{W, Z\}$ meet one S_i. Our method of proof is inductive, and applicable when $n \geq 8$. Accordingly, $\mathcal{F}$ has many $(n-2)$-transversals, and we show that certain of them yield the existence of either an $(n-1)$-transversal or a separating $(n-2)$-transversal. In the latter case, we apply the following from [5]:

LEMMA: *If $n = |\mathcal{F}| \geq 8$ and $\mathcal{F}$ has a separating $(n-2)$-transversal, then $\mathcal{F}$ has the property $T - 1$.*

Since this lemma is the cornerstone of our argument, we give a brief outline of its proof.

Assume that H is an $(n-2)$-transversal of $\mathcal{F}$ that strictly separates U and V, $\{U, V\} \subset \mathcal{F}$. Without loss of generality, we may assume that H is horizontal. Let L and L' be the tangential separators for U and V. Note that some vertical line that meets $[U, V]$ is disjoint from every $B_i \in \mathcal{H}$, where $\mathcal{H} = \{B_i \in \mathcal{F} \backslash \{U, V\} \colon B_i \cap [U, V] = \emptyset\}$. Hence, the transversal for $\{U, V, B_i\}$ has either positive slope or negative slope. Assume the former. Then the property $T(4)$ for $\{U, V, B_i, B_j\}$ yields only positive-sloped transversals for each $B_j \in \mathcal{H}$. Thus each $B_i \in \mathcal{H}$ meets L or L' depending upon which has smaller positive slope.

Next, we assume that H is an $(n-2)$-transversal that strictly separates U and V, and there is a line orthogonal to H that intersects U and V. We prove in [5] that there is an $(n-1)$-transversal of $\mathcal{F}$ or an $(n-2)$-transversal H' and $\{U', V'\} \subset \mathcal{F}$ such that H' strictly separates U' and V', and at most two elements of $\mathcal{F} \setminus \{U', V'\}$ meet the line segment $H' \cap [U', V']$.

In view of this result, it is sufficient to prove that $\mathcal{F}$ has an $(n-1)$-transversal if no line orthogonal to H intersects both U and V, or if at most two elements of $\mathcal{F} \setminus \{U, V\}$ meet the line segment $H \cap [U, V]$. The former can be proved by a simple argument while the latter requires a case analysis which depends on the relative positions of the partial transversal H and the elements of $\mathcal{F} \setminus \{U, V\}$. For more details, see [5].

3. $T(4)$ implies $T - 1$

Main Theorem: *Let $\mathcal{F}$ be a finite family of mutually disjoint unit disks with the property $T(4)$. Then $\mathcal{F}$ has the property $T - 1$.*

Before embarking on the proof of the Main Theorem, we give a brief outline of the argument in order to help the reader understand the details. First of all, we choose two special $(n-2)$-transversals of the family $\mathcal{F}$, one of which avoids W and another disk X, the other avoids Z and a disk Y in such a way that they are not separating $(n-2)$-transversals. The proof has two major parts. In the first part, we consider the case when these two chosen $(n-2)$-transversals have slopes of the same sign, say negative, cf. Figure 2, and assume that the $(n-2)$-transversal avoiding W and X is in limit position in such a way that its distance from $W \cup X$ is minimal. Under these assumptions, using a delicate case analysis, we determine the positions of the other elements of $\mathcal{F}$ and show that either the family $\mathcal{F}$ has an $(n-1)$-transversal or there is a separating $(n-2)$-transversal. In the second part of the proof, we assume that the two $(n-2)$-transversals have slopes of different signs, cf. Figure 3. We choose the $(n-2)$-transversals again in limit positions such that they determine a minimal acute angle. Now, this configuration gives rise to three possible combinations of the relative positions of the disks X and Y according to which S_i they intersect. In each case, with a detailed case analysis, we establish the possible positions of the other members of $\mathcal{F}$ and either directly find an $(n-1)$-transversal or a separating $(n-2)$-transversal.

Proof. Clearly, we may assume that each S_i meets at least three disks of $\mathcal{F} \setminus \{W, Z\}$. We let $\tilde{S}_i = S_i \cap [W, Z]$, and assume that the S_i's are horizontal; (cf. Figure 1.)

By induction, $\mathcal{F} \setminus \{W\}$ has an $(n-2)$-transversal that is disjoint from, say, $X \in \mathcal{F} \setminus \{W\}$. Let T_{wx} denote such a line. Similarly, there is an $(n-2)$-transversal of $\mathcal{F}$ that is disjoint from Z and a disk $Y \in \mathcal{F} \setminus \{Z\}$. Let T_{zy} denote such a line. By the Lemma, we may assume that no T_{wx} and no T_{zy} is a separating $(n-2)$-transversal. Let $\mathcal{F}' = \mathcal{F} \setminus \{X, Y, Z, W\}$. Clearly, we may assume that no disk of $\mathcal{F}'$ meets both S_1 and S_2. If some disk C in $F \setminus \{W, Z\}$ meets both S_1 and S_2, then choosing an $(n-2)$-transversal L that does not meet C readily yields that L does not meet both W and Z. Assume that L does not meet W. Then clearly C is the disk of this type that is closest to W.

Thus, there are at most two such disks of this type which we may assume are X and Y.

We assume that all T_{wx} and all T_{zy} have slopes of the same sign. Let $w(z)$ denote the point in $\mathrm{int}(K)$ that is contained in $T_{zy} \cap \mathrm{bd}(W)$ ($T_{wx} \cap \mathrm{bd}(Z)$), and let N be the line through w and z. We consider a disk A with the property that $S_2 \cap A \neq \emptyset$ and $N \cap A = \emptyset$. Then either $T_{wx} \cap A = \emptyset$ and $A = X$, or $T_{wx} \cap A \neq \emptyset$ and $T_{wx} \cap Z \subset [T_{wx} \cap A, T_{wx} \cap U]$ for any $U \in (\mathcal{F}' \cup \{Y\}) \setminus \{A\}$. Similarly, a disk B that meets S_1 and that is disjoint from N has the property that either $B \cap T_{zy} = \emptyset$ and $B = Y$ or $T_{zy} \cap W \subset [T_{zy} \cap B, T_{zy} \cap U]$ for any $U \in (\mathcal{F}' \cup \{X\}) \setminus \{B\}$.

We note that if there are at most one such A and at most one such B for some T_{wx} and T_{zy}, then N is either an $(n-1)$-transversal or a separating $(n-2)$-transversal. Accordingly, we may assume that there are two disks disjoint from N and meeting S_2. These disks are necessarily X, a unique $A_2 \in \mathcal{F}' \cup \{Y\}$, and $T_{wx} \cap Z \subset [T_{wx} \cap A_2, T_{wx} \cap U]$ for all $U \in (\mathcal{F}' \cup \{Y\}) \setminus \{A_2\}$ and for all T_{wx}.

We choose now a T_{wx} in a limit position; that is, we assume that T_{wx} is closest to $W \cup X$. If T_{wx} is a tangential separator of Z and A_2, then it follows from $T(4)$ and the property of T_{wx} that all elements of $\mathcal{F} \setminus \{X\}$ meet the other tangential separator of Z and A_2. Otherwise, T_{wx} supports from below some $U \neq Z$ that meets S_1, and T_{wx} is either a tangential separator of U and A_2 or a line of common support for U and Z. In either case, $M \cap Z \subset [M \cap A_2, M \cap U]$ for any transversal M of $\{A_2, U, Z\}$. Then $T(4)$ for $\{A_2, U, Z, W\}$ yields that one such oriented M meets the disks in the order (w, u, z, a_2) or (u, w, z, a_2); both of which contradict the limit property of T_{wx}.

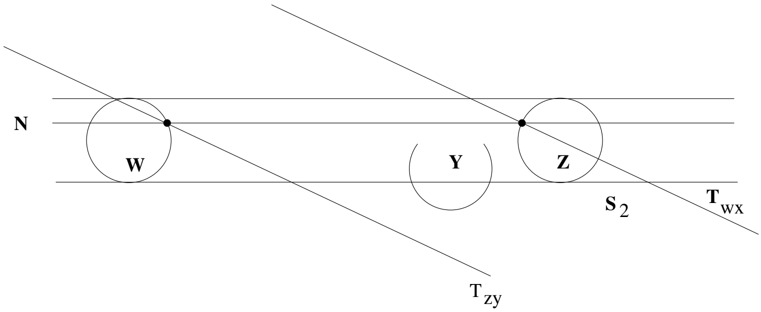

Figure 2

Let some T_{wx} have slope of the same sign, say positive, and let some T_{zy} have negative slope. We assume the following:

There is no $(n-1)$-transversal of $\mathcal{F}$ that meets both W and Z.

Hence, there is an $L = T_{wx}$ and an $M = T_{zy}$ that determine a minimum acute angle α. Since $|\mathcal{F}'| \geq 4$ and L and M are transversals of $\mathcal{F}'$, it readily follows that $\alpha < 60°$.

Let $L \cap M = \{p\}$ and denote by L^+ (L^-) and M^+ (M^-) the open half-lines of L and M, respectively, that are right (left) of p. Let S_p denote the line through p parallel to S_i, cf. Figure 3.

Next, let $U \in \mathcal{F}'$. We say that U is **vertical** if U meets L^+ and M^+, or L^- and M^-, and that U is **horizontal** if $L \cap U \subset L^+$ and $M \cap U \subset M^-$, or $L \cap U \subset L^-$ and $M \cap U \subset M^+$.

We note that there is a supporting line S_u of U that is parallel to S_i and meets both W and Z.

Finally, we denote also by $A^*, \tilde{A}, A_1, A_2, \ldots$ ($B^*, \tilde{B}, B_1, B_2, \ldots$) the elements of $\mathcal{F}'$ that intersect S_1 (S_2). We let $S_u = S_a^*$ if $U = A^*$, and define $\tilde{S}_a, S_b^*$ and $\tilde{S}_b$ in similar manner.

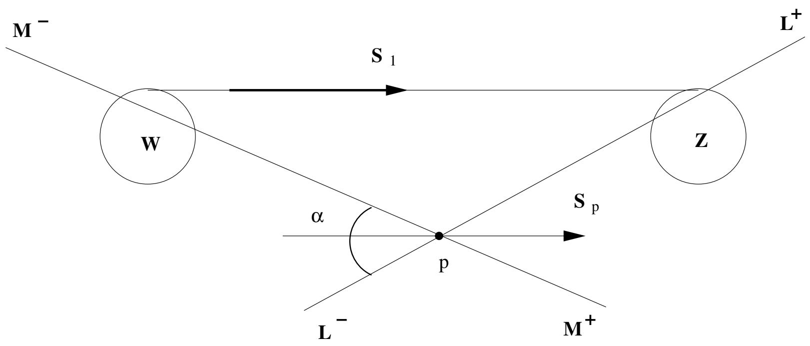

Figure 3

Observations:

(1) Each $U \in \mathcal{F}'$ is either vertical or horizontal.
(2) If $U \in \mathcal{F}'$ is vertical, then U meets S_p.
(3) $\alpha < 60°$ implies that at most one A_i, denoted by A^*, and at most one B_j, denoted by B^*, are horizontal.

(4) If A^* and B^* exist, then S_a^* is a transversal of $\mathcal{F} \setminus \{X, Y, B^*\}$, and S_b^* is a transversal of $\mathcal{F} \setminus \{X, Y, A^*\}$.

(5) If A^* exists but B^* does not, then either S_a^* is a transversal of $\mathcal{F}\setminus\{X, Y\}$ or there is at most one B_j, denoted by $\tilde{B}$, such that $\tilde{B}$ meets either M^+ and $[p, L^+ \cap A^*]$, or L^- and $[p, M^-, A^*]$.

(6) If A^* and $\tilde{B}$ exist, then S_a^* is a transversal of $\mathcal{F} \setminus \{X, Y, \tilde{B}\}$, and $\tilde{S}_b$ is a transversal of $\mathcal{F} \setminus \{X, Y, A^*\}$.

We note that there are analogous statements to (5) and (6) with A and B interchanged.

We consider now the possible positions of X and Y, and observe that slope $M < 0 <$ slope L implies that $\tilde{S}_2$ meets M and L.

Case I: $\tilde{S}_i \cap X \neq \emptyset \neq \tilde{S}_j \cap Y$, $i \neq j$

We may assume that, say, $\tilde{S}_1 \cap X \neq \emptyset \neq \tilde{S}_2 \cap Y$. It is easy to check that $p \notin [W, Z]$ implies that at most two disks of $\mathcal{F}\setminus\{W, Z\}$ meet $\tilde{S}_1$. Then $p \in [W, Z]$, and S_p intersects W, Z and Y. Thus, if neither A^* nor B^* exist, then S_p is a transversal of $\mathcal{F} \setminus \{X\}$ by (2). This contradicts that there is no $(n-1)$-transversal of meeting both W and Z.

If A^* exists, then $\alpha < 60°$ implies that Y meets S_a^*, and we may assume that either B^* or $\tilde{B}$ exists, and that $S_a^* \cap (X \cup U) = \emptyset$ for $U \in \{B^*, \tilde{B}\}$. We note that S_a^* separates X and U, and that S_a^* is a transversal of $\mathcal{F} \setminus \{X, U\}$ by (4) or (6).

Let B^* exist. Then S_b^* meets Y, and we may assume from above that $\tilde{A}$ exists. By (6), we obtain now that $\tilde{S}_a$ is a separating transversal of $\mathcal{F} \setminus \{X, B^*\}$.

Case II: $\tilde{S}_2 \cap X \neq \emptyset \neq \tilde{S}_2 \cap Y$

We note that $X = Y$ implies that $p \notin [W, Z]$ and that no $U \in \mathcal{F}'$ intersects S_1; that is, S_2 is a transversal of $\mathcal{F}$.

Let $X \neq Y$. From above, we may assume that $p \in [W, Z]$. Then $p \in [X, Y]$ and S_p intersects X, Y, W and Z. By (2), S_p is a transversal of $\mathcal{F} \setminus \{A, B\}$, where $A \in \{A^*, \tilde{A}\}$ and $B \in \{B^*, \tilde{B}\}$. We may assume that such A and B exist, and that $S_p \cap (A \cup B) = \emptyset$. Clearly, S_p separates A and B.

Case III: $\tilde{S}_1 \cap X \neq \emptyset \neq \tilde{S}_1 \cap Y$

First, if $X = Y$, then we deduce from the Observations that $p \in [W, Z]$. We argue now as in Case II in regards to the possible existence of A^*, B^* and $\tilde{A}$, and thus obtain a separating $(n-2)$-transversal or an $(n-1)$-transversal in each case. Let $X \neq Y$. Then, in view of Case II, we may assume that $p \notin [X, Y]$.

Since α is minimal, M (L) is a separating tangent of disks U and V (C and D) such that $\tilde{S}_1 \cap U \neq \emptyset \neq \tilde{S}_1 \cap C$ and $\tilde{S}_2 \cap V \neq \emptyset \neq \tilde{S}_2 \cap D$; cf. Figure 4 for the relative positions of p with respect to U and V (C and D).

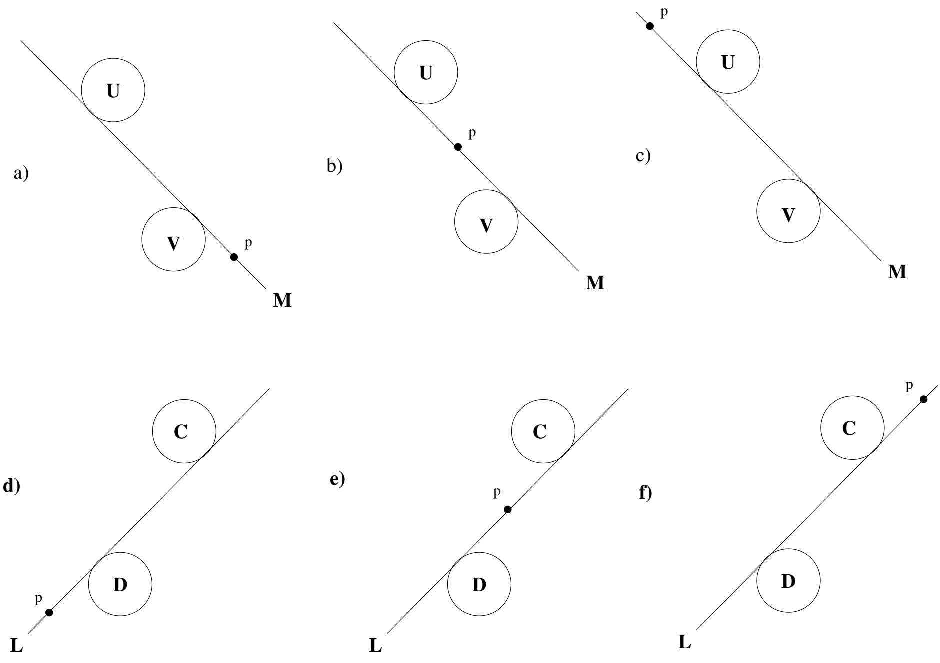

Figure 4

Observations:

(7) If $U = C$, then $U = A^* = C$, and if $V = D$ then $V = B^* = D$.
(8) $V = B^*$ in case of b) or c), and $D = B^*$ in case of e) or f).
(9) $U \in \{X, A^*\}$ in case of a) or b), and $C \in \{Y, A^*\}$ in case of d) or e).
(10) $U = A^*$ and a) imply that $V = \tilde{B}$, and $C = A^*$ and d) imply that $D = \tilde{B}$.

It is easy to check that if $U = C$ and $V = D$, then $\{W, A^*, B^*, Z\}$ has no transversal. We note that $\{U, C\} \cap \{X, Y\} = \emptyset$ implies that L meets U, M

meets V, and thus, $U = C$. From (7) to (10); $U = C$ yields that $V \neq D$, $U = A^* = C$ and a) or b).

If a), then $V = \tilde{B}$, B^* does not exist and d). But $C = A^*$ and d) imply $D = \tilde{B} = V$; a contradiction. If b), then $V = B^*$, $\tilde{B}$ does not exist and d). Again, $C = A^*$ and d) imply $D = \tilde{B}$; a contradiction. Thus $U \neq C$ and $\{U, C\} \cap \{X, Y\} \neq \emptyset$. In view of the symmetry between L and M, we may assume that, say, $U = X$, cf. Figure 5.

Let $\overline{M}$ denote the other tangential separator of X and V. We note that

$$V \cap [X, Z] = \emptyset = Z \cap [V, X].$$

Thus, $T(3)$ for $\{V, X, Z\}$ implies that $\overline{M}$ meets Z, and $T(4)$ for $\{W, V, X, Z\}$ implies that $\overline{M}$ meets W. From this, it follows that

$$X \cap [V, Z] \neq \emptyset \neq V \cap [W, X].$$

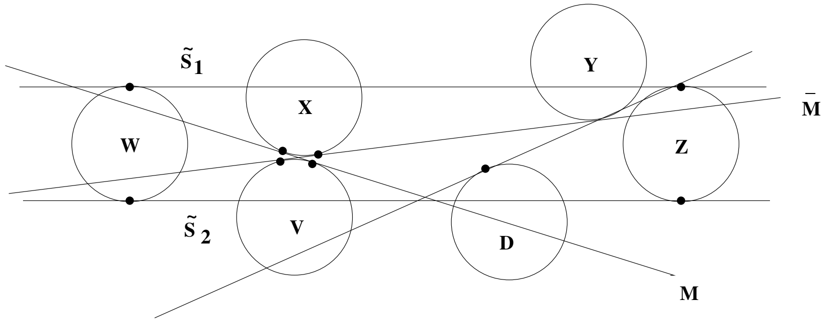

Figure 5

Since $L = T_{wx}$ separates D and $[W, X]$, if follows from $V \cap [W, X] \neq \emptyset$ that $D \neq V$. Thus, (8) implies that M satisfies a) or L satisfies d). It is now easy to check that $\overline{M}$ separates $\tilde{S}_1$ and p, and thus, it follows from the fact that each $A_i \in \mathcal{F}'$ intersects S_1, L and M, that $A_i \cap \overline{M} \neq \emptyset$. Next, $p \in [V, D]$ follows from the fact that both L and M meet V and D, and thus, $\overline{M}$ intersects each $B_j \in \mathcal{F}'$ that is disjoint from $[V, Z]$.

Finally, we consider the disks B_j that are distinct from V and meet both $\tilde{S}_2$ and $[V, Z]$. We note that if $B_j \cap [X, V] = \emptyset$, then $T(4)$ for $\{X, V, B_j, Z\}$ yields that $\overline{M}$ intersects B_j. Thus, $\overline{M}$ intersects all members of $\mathcal{F}'$ with the possible exception of Y and those $B_j \neq V$ that meet $[X, V]$. It is clear that there is at most one such B_j, say, $\overline{B}$, and if $\overline{M} \cap \overline{B} = \emptyset$, then $\overline{M}$ separates Y and $\overline{B}$. ∎

Note, that while it is possible that our proof is applicable to any planar family of disjoint translates of an arbitrary oval, the dependence of our argument in the case of seven disjoints disks upon a specific configuration (cf. Figure 4 of [4]) of eight disks seems to indicate that the possibility is remote.

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